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ON SPONTANEOUS SWIRLING IN AXISYMMETRIC FLOWS

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The problem of spontaneous swirling was considered in [1–4] and is as follows: would the rotary motion arise with no external supply source of rotation, i.e., under conditions where the flow without rotation is known to be.

This problem can be stated in more detail as follows. Assume that, in some region enclosed by a surface of revolution, there is an axisymmetric flow supported by the appropriate velocity distribution at the boundary and by volume forces. The flow region may be both singly and multiply connected. The boundary is conventionally divided into two segments so that the parameter vanishing on one segment is the rotational velocity, while on the other segment either both the rotational component of the tangential stress and the normal velocity are simultaneously equal to zero or the former is equal to zero and the latter is directed outside the volume under considerations.

It should be emphasized that in this article we are not concerned with the question of the feasibility of such boundary conditions in experiment. However, from a purely mathematical point of view, these conditions are fully acceptable and correct.

Thus, is it possible, with the given boundary conditions satisfied, that, due to the loss of stability, the initial flow bifurcates into the rotary flow (not necessarily rotationally symmetric) such that the circulation on the circumference that lies in the plane normal to the symmetry axis containing the circumference center is different from zero, at least in some part of the region?

At present there is a hypothesis on the possibility of spontaneous swirling. The phenomenon was named "autorotation" or "vortical dynamo" [1–3]. The authors of the hypothesis think that the existence of the spontaneous swirling has been proved. However, as was shown in [4], the available examples of "autorotation" do not satisfy the above conditions providing for close control over the angular momentum flux.

Thus, in [1], the problem on the flow periodic along the z-axis between two planes $z = 0$, $z = L$ is examined, with nonzero angular momentum flux inflow at $z = 0$. We may see further that the steadiness of the solution found (in the relevant frame of reference) points to the possibility of a self-maintained rotationally symmetric, on average, flow due to the counter-gradient angular momentum flux. However, the following question remains open: would such a flow arise over the bounded region or half-space $z > 0$ if the condition $\Gamma = 0$ is specified on the plane $z = 0$, and by doing so, would the inward angular momentum flux be eliminated? Only with this condition satisfied is it possible to speak about the occurrence of spontaneous swirling.

A similar problem exists for the occurrence of a spontaneous cross (normal) flux in the case of an initial two-dimensional parallel flow.

Assume that in an arbitrary cylinder with a generatrix parallel to the z axis, there is a plane flow $\mathbf{V} = (U(x,y), V(x,y), 0)$ maintained by an appropriate velocity distribution on the boundary and by volume forces.

If we specify the condition $V_z = W = 0$ on the segment l' of the region boundary l and demand that $\partial W/\partial n$ vanish on the remainder l'' and the non-inflow condition $\mathbf{Vn} \geqslant 0$ be satisfied ($\mathbf{n}$ is the outward normal to the cylinder surface), it is unknown whether or not a flow with nonzero velocity component $W(x,y,z)$

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appears such that

$$\langle W(x,y)\rangle = \frac{1}{2L}\int\limits_{-L}^{L} W(x,y,z)\,dz \not\equiv 0,$$

and the pressure $p(x,y,z)$ satisfies the condition

$$\frac{1}{2L}[p(x,y,L) - p(x,y,-L)] \to 0, \quad \text{where} \quad L \to \infty.$$

A problem of this kind, i.e., flow of a conductive fluid between two planes in the presence of a magnetic field, was considered in [5]; it has been shown that upon imposing disturbances as periodical waves at some angle to the main flow, a bifurcation to the flow with $\langle W(x,y)\rangle \neq 0$ arises, provided that $W = 0$ on the walls. The defect of this example is that there is a flow of the zth momentum component from infinity. The proof would be an example where the inward angular momentum flux is eliminated, i.e., investigation of the corresponding problem for a bounded or half-bounded region.

To narrow the region of search, it is of interest to consider the corresponding problems with z-independent U, V, W. Such problems may be considered as a plane analog of the rotationally symmetric flows.

In this case the equation for W takes the form (the fluid is assumed to be incompressible and viscous)

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} = \nu\Big(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\Big), \tag{1}$$

with U and V satisfying the continuity equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0,$$

otherwise being arbitrary.

It is easy to see that under the boundary conditions accepted earlier the spontaneous occurrence of the cross flow (i.e., $W(x,y) \neq 0$) is impossible. Indeed, on multiplying Eq. (1) by W and integrating over the cylinder section D, we obtain

$$\frac{\partial}{\partial t}\int\limits_D W^2(x,y)\,dx\,dy = -\oint\limits_l (\mathbf{Vn})W^2\,dl + 2\nu\oint\limits_l W\frac{\partial W}{\partial n}\,dl - 2\nu\int\limits_D \Big[\Big(\frac{\partial W}{\partial x}\Big)^2 + \Big(\frac{\partial W}{\partial y}\Big)^2\Big]\,dx\,dy. \tag{2}$$

In view of the boundary conditions, the second term on the right-hand side of the equation vanishes and the first term is nonnegative on l'' and equals zero on l'.

Hence, for an arbitrary initial disturbance $W_0(x,y)$ the cross flow is damped out. This result is also valid for a compressible gas, arbitrary coordinate dependence of the viscosity coefficient ν, and time-dependent two-dimensional flow. In this case, instead of (2), we have the equation

$$\frac{\partial}{\partial t}\int\limits_D \rho W^2\,dx\,dy = -\oint\limits_l \rho(\mathbf{Vn})W^2\,dl + 2\oint\limits_l \nu\rho W\frac{\partial W}{\partial n}\,dl - \int\limits_D \nu\rho\Big[\Big(\frac{\partial W}{\partial x}\Big)^2 + \Big(\frac{\partial W}{\partial y}\Big)^2\Big]\,dx\,dy,$$

which leads to the same conclusion that the cross flow is absent.

In [4], it has been proved that under the above boundary conditions steady axisymmetric flows with the circulation $\Gamma \neq 0$ are impossible. Now we show that any rotationally symmetric flow with $\Gamma \neq 0$ at the initial time approaches the axisymmetric flow with $\Gamma = 0$.

In the time-dependent flow of a viscous compressible fluid, the continuity equation and the equation for Γ are of the forms

$$\frac{\partial}{\partial t}(\rho r) + \frac{\partial}{\partial r}(\rho r u) + \frac{\partial}{\partial z}(\rho r w) = 0,$$

$$\frac{\partial}{\partial t}(\rho r\Gamma) + \frac{\partial}{\partial r}(\rho ru\Gamma) + \frac{\partial}{\partial z}(\rho rw\Gamma) = \frac{\partial}{\partial r}\left[\rho\nu\left(r\frac{\partial\Gamma}{\partial r} - 2\Gamma\right)\right] + \frac{\partial}{\partial z}\left[\rho\nu r\frac{\partial\Gamma}{\partial z}\right], \tag{3}$$

where ρ , u, and w can be time-dependent. Integrating the second equation of (3) over the section D cut by the meridional plane yields

$$\frac{\partial}{\partial t}\int\limits_D \rho r\Gamma\, dr\, dz = -\oint\limits_l \rho(un_r + vn_z)\Gamma\, dl + \oint\limits_l \rho\nu r\frac{\partial\Gamma}{\partial n}\, dl. \tag{4}$$

If the value of Γ is positive (negative) over the entire region D, then the right-hand side of Eq. (4) is nonpositive (nonnegative) since $un_r + vn_z \geqslant 0$ and $\partial\Gamma/\partial n \leqslant 0$ and the rotation is damped out. If Γ changes sign in the flow region, there is a line $\Gamma = 0$ within the region D. In this case the integration is carried out over the region where the Γ values are positive (negative). Since Γ vanishes on the additional boundary, we obtain an equality similar to (4), and it follows that the rotation over the subregion and consequently over the entire region is damped out.

As an example of the unbounded region we consider the Burgers turbulent vortex, assuming that the turbulent viscosity $\nu_* = \nu_*(r,t) > 0$ depends only on r and t. In this case there is the solution

$$v_r = -a(t)r, \quad v_z = 2a(t)z, \quad \Gamma = \Gamma(r,t),$$

with Γ satisfying the equation

$$\frac{\partial}{\partial t}(r^3\omega) - ar^2\frac{\partial}{\partial r}(r^2\omega) = \frac{\partial}{\partial r}\left(\nu_* r^3\frac{\partial\omega}{\partial r}\right), \tag{5}$$

when $\omega = \Gamma/r^2$; $\omega(0,t) = \omega(t)$; $\omega r^3 \to O(r^{-(1+\epsilon)})$ as $r \to \infty$.

Let $\omega = \omega_0(r)$ at the initial moment, and $\omega_0 \to 0$ as $r \to \infty$ so that

$$\int\limits_0^\infty r^3\omega\, dr = M_0 < \infty.$$

Then, multiplying (5) by ω, we obtain

$$\frac{\partial}{\partial t}(r^3\omega^2/2) - \frac{\partial}{\partial r}(ar^4\omega^2/2) = \frac{\partial}{\partial r}\left(\nu_*(r,t)r^3\frac{\partial\omega}{\partial r}\right) - \nu_* r^3\left(\frac{\partial\omega}{\partial r}\right)^2. \tag{6}$$

Integrating (6) with respect to r from 0 to ∞ subject to the boundary conditions, we find

$$\frac{\partial}{\partial t}\int\limits_0^\infty (r^3\omega^2/2)\, dr = -\int\limits_0^\infty \nu_* r^3\left(\frac{\partial\omega}{\partial r}\right)^2 dr,$$

from which it follows that $\omega \to 0$, i.e., spontaneous swirling is absent.

Also, it is easy to see that in the general case where some part of the boundary is located at infinity no spontaneous swirling occurs under the strict conditions mentioned above (no inward z-component flux of the angular momentum). In particular, given the condition $\Gamma = 0$ on the plane $z = 0$, spontaneous swirling in a half-bounded space is also impossible.

Thus, no bifurcation of the axisymmetric flow into the rotational-symmetric flow takes place for a compressible fluid with a variable viscosity coefficient. It also follows that in the models that describe turbulence by the introduction of a turbulent viscosity coefficient greater than zero, spontaneous swirling cannot arise.

If the distribution is not axisymmetric, then integrating over φ, and taking into account that all the quantities are periodically dependent on φ, we obtain the following expression for an incompressible viscous fluid with a constant viscosity coefficient:

$$\frac{\partial}{\partial t}(r\langle\Gamma\rangle) + \frac{\partial}{\partial r}(r\langle u\Gamma\rangle) + \frac{\partial}{\partial z}(r\langle w\Gamma\rangle) = \frac{\partial}{\partial r}\left[\nu\left(r\frac{\partial\langle\Gamma\rangle}{\partial r} - 2\langle\Gamma\rangle\right)\right] + \frac{\partial}{\partial z}\left(r\nu\frac{\partial\langle\Gamma\rangle}{\partial z}\right) \tag{7}$$

(angle brackets designate averaging over φ).

Now integrating (7) over the section D in the meridional plane, we find

$$\frac{\partial}{\partial t}\int_D r\langle\Gamma\rangle\,dr\,dz = -\oint_l (\langle u\Gamma\rangle n_r + \langle w\Gamma\rangle n_z)\,dl + \oint_l r\nu\,\frac{\partial\langle\Gamma\rangle}{\partial n}\,dl. \qquad (8)$$

Hence it follows that within the flow region $\langle\Gamma\rangle$ cannot be of constant sign since in this case relationship (8), subject to the boundary conditions, gives

$$\frac{\partial}{\partial t}\int_D r\langle\Gamma\rangle\,dr\,dz \leqslant 0.$$

Thus, the rotation is enhanced only if $\langle\Gamma\rangle$ changes sign over the region D. If we now separate out a region with $\langle\Gamma\rangle$ of fixed sign, then on the additional boundary with $\langle\Gamma\rangle = 0$ the flow $\langle u\Gamma\rangle n_r + \langle w\Gamma\rangle n_z$ may be different from zero, but for the rotation to intensify the flow should be directed from a region with small $\langle\Gamma\rangle$ values to a region with great ones, i.e., a countergradient flow of $\langle\Gamma\rangle$ has to arise and hence the intensification of the rotation is accounted for by the division of the angular momentum between different parts of the flow.

Sometimes, in hydrodynamics flows of this type are called phenomena with a "negative" viscosity. An example of such flows is the well-known Rank effect, where the flows of a low and high temperature are separated. However, at present there is no sufficiently strong and convincing explanation for the angular momentum division mechanism.

The flow considered in [1] cannot be treated as an example of spontaneous swirling but shows the possibility of the occurrence of a countergradient angular momentum flux in nonaxisymmetric swirling flows.

The question of the existence of such a mechanism (and, consequently, the spontaneous swirling) under the conditions stated above remains open.

It is of interest to consider the problem of spontaneous swirling in magnetohydrodynamic flows. The problem on the occurrence of a spontaneous swirling flow between two coaxial pipes in the presence of a magnetic field as a result of bifurcation of the main axisymmetric flow and its two-dimensional analog was examined in [5, 6].

Since in both cases the conditions of the absence of the inward flows of the cross momentum and of the zth angular momentum component are not satisfied, these problems cannot be considered examples of the spontaneous occurrence of cross flow or swirling. The two-dimensional analog and axisymmetric flow of a viscous compressible conducting fluid are investigated below. The equations governing such flows have the following forms (in conventional notations):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} - \frac{1}{4\pi\rho}(\mathbf{H}\times\ \mathrm{rot}\ \mathbf{H}) + \mathbf{f},$$

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{rot}\ (\mathbf{v}\times\mathbf{H}) + \nu_m\Delta\mathbf{H}, \quad \mathrm{div}\ \mathbf{v} = 0, \quad \mathrm{div}\ \mathbf{H} = 0.$$

Here $\mathbf{f} = (f_x, f_y, 0)$ for the two-dimensional case and $\mathbf{f} = (f_r, 0, f_z)$ in the axisymmetric flow; $\nu_m = c^2/4\pi\sigma$.

At first we consider the two-dimensional analog: the possibility of appearance of the cross flow with velocity $V_z = W(x,y)$ in the preceding statement of the problem. In addition, we assume that the cylinder walls are nonpenetrating and superconducting so that the nonpenetration condition $\mathbf{vn} = 0$ and condition $\mathbf{Hn} = 0$ are satisfied on the region boundary. In addition, from the requirement that the electric field component tangential to the walls vanish, it follows that $\partial H_z/\partial n = 0$.

Let, over the region D enclosed by the boundary l (the cylinder section cut by the plane normal to the cylinder generatrix), there be a flow

$$\mathbf{v} = (u(x,y,t),\ v(x,y,t), 0)$$

and the magnetic field

$$\mathbf{H} = (H_x(x,y,t),\ H_y(x,y,t),\ H_z(x,y,t)).$$

At the moment $t = 0$ the disturbance $V_z = W = W_0(x,y)$ is produced. We show that $w(x,y,t) \to 0$ as $t \to \infty$.

The equations describing the behavior of H_z and w have the forms

$$\frac{\partial H_z}{\partial t}+u\frac{\partial H_z}{\partial x}+v\frac{\partial H_z}{\partial y}=H_x\frac{\partial w}{\partial x}+H_y\frac{\partial w}{\partial y}+\nu_m\Big(\frac{\partial^2 H_z}{\partial x^2}+\frac{\partial^2 H_z}{\partial y^2}\Big); \tag{9}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial t}=\frac{1}{4\pi\rho}\Big(H_x\frac{\partial H_z}{\partial x}+H_y\frac{\partial H_z}{\partial y}\Big)+\nu\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\Big). \tag{10}$$

Moreover,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}=0. \tag{11}$$

Multiplying Eq. (9) by $H_z/4\pi$ and (10) by ρw and adding the relationships obtained, we find

$$\frac{\partial \varepsilon}{\partial t}+\frac{\partial}{\partial x}(u\varepsilon)+\frac{\partial}{\partial y}(v\varepsilon)=\frac{1}{4\pi}\Big(\frac{\partial}{\partial x}(H_xH_zw)+\frac{\partial}{\partial y}(H_yH_zw)\Big)+\nu_m\Big(\frac{\partial}{\partial x}\Big[H_z\frac{\partial H_z}{\partial x}\Big]+\frac{\partial}{\partial y}\Big[H_z\frac{\partial H_z}{\partial y}\Big]\Big)$$

$$-\nu_m\Big[\Big(\frac{\partial H_z}{\partial x}\Big)^2+\Big(\frac{\partial H_z}{\partial y}\Big)^2\Big]+\rho\nu\Big(\frac{\partial}{\partial x}\Big[w\frac{\partial w}{\partial x}\Big]+\frac{\partial}{\partial y}\Big[w\frac{\partial w}{\partial y}\Big]\Big)+\rho\nu\Big[\Big(\frac{\partial w}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial y}\Big)^2\Big], \tag{12}$$

where $\varepsilon=\rho w^2/2+H_z^2/8\pi$.

Integration of (12) over the region D yields

$$\frac{\partial}{\partial t}\int_D \varepsilon\,dx\,dy=-\oint_l(\mathbf{Vn})\varepsilon\,dl+\frac{1}{4\pi}\oint_l(\mathbf{Hn})H_zw\,dl+\oint_l\Big(\nu_mH_z\frac{\partial H_z}{\partial n}+\nu w\frac{\partial w}{\partial n}\Big)dl$$

$$-\int_D\Big(\nu_m\Big[\Big(\frac{\partial H_z}{\partial x}\Big)^2+\Big(\frac{\partial H_z}{\partial y}\Big)^2\Big]+\nu\Big[\Big(\frac{\partial w}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial y}\Big)^2\Big]\Big)dx\,dy. \tag{13}$$

From relationship (13), in view of the boundary conditions, it follows that $w(x,y,t)\to 0$ if the condition $w=0$ is fulfilled in the infinitesimal part of l'.

Thus, under the conditions considered, the spontaneous occurrence of the cross flow is impossible.

For the axisymmetric geometry, it is imposssible to obtain a general result such as that for the two-dimensional case. The fact is that in this case there may be flows with a swirling maintained by electromagnetic forces, as will be shown below. However, such flows cannot be treated as examples of spontaneous swirling, because there is an explicit rotary source, namely, the magnetic field.

In this connection, we consider the steady flow of a viscous incompressible superconducting fluid. The equations have the following forms:

for the circulation,

$$rU\frac{\partial\Gamma}{\partial r}+rW\frac{\partial\Gamma}{\partial z}=\frac{\partial}{\partial r}\nu\Big(r\frac{\partial\Gamma}{\partial r}-2\Gamma\Big)+\frac{\partial}{\partial z}\Big(\nu r\frac{\partial\Gamma}{\partial z}\Big)+\frac{1}{4\pi\rho}\Big(rH_r\frac{\partial rH_\varphi}{\partial r}+rH_z\frac{\partial rH_\varphi}{\partial z}\Big), \tag{14}$$

$$V_r=U,\quad V_\varphi=V,\quad V_z=W,\quad \Gamma=rV_\varphi;$$

for the magnetic field:

$$VH_z-WH_\varphi=\frac{\partial\Phi}{\partial r}; \tag{15}$$

$$WH_r-UH_z=\frac{1}{r}\frac{\partial\Phi}{\partial\varphi}=\frac{C}{r}; \tag{16}$$

$$UH_\varphi-VH_r=\frac{\partial\Phi}{\partial z}, \tag{17}$$

where C is an arbitrary constant, which, in the given statement of the problem, is equal to zero in view of the boundary conditions ($\mathbf{vn}=0$, $\mathbf{Hn}=0$).

Taking into account that

$$\frac{\partial rU}{\partial r}+\frac{\partial rW}{\partial z}=0,\quad \frac{\partial rH_r}{\partial r}+\frac{\partial rH_z}{\partial z}=0,$$

from (15)–(17) we obtain

$$U=-\frac{1}{r}\frac{\partial\Psi}{\partial z},\quad W=\frac{1}{r}\frac{\partial\Psi}{\partial r},\quad H_z=\frac{1}{r}\lambda(\Psi)\frac{\partial\Psi}{\partial r},\quad H_r=-\frac{1}{r}\lambda(\Psi)\frac{\partial\Psi}{\partial z},\quad H_\varphi=rf(\Psi)+\lambda(\Psi)V_\varphi. \quad (18)$$

Substituting (18) into (14), we find

$$\frac{\partial}{\partial r}\Big[\Big(1-\frac{\lambda^2}{4\pi\rho}\Big)rU\Gamma\Big]+\frac{\partial}{\partial z}\Big[\Big(1-\frac{\lambda^2}{4\pi\rho}\Big)rW\Gamma\Big]+\frac{\lambda fr}{2\pi\rho}r\frac{\partial\Psi}{\partial z}=\frac{\partial}{\partial r}\Big[\nu\Big(r\frac{\partial\Gamma}{\partial r}-2\Gamma\Big)\Big]+\frac{\partial}{\partial z}\Big[\rho\nu r\frac{\partial\Gamma}{\partial z}\Big].$$

If $f(\Psi)=0$, one can easily see that, under the ordinary boundary conditions ($\mathbf{Vn}=0$, either Γ or τ_φ are equal to zero on the boundary), there are no solutions with $\Gamma\not\equiv 0$, i.e., swirling is absent.

If $f(\Psi)\not\equiv 0$, it is obvious that there exist some flows with $\Gamma\not\equiv 0$ defined by the specific type of the functions $\lambda(\Psi)$ and $f(\Psi)$. As mentioned above, such flows cannot be considered as examples of spontaneous swirling. On the contrary, in investigating the question on the possibility of occurrence of spontaneous swirling such flows must be eliminated. For this, in the initial state, H_φ must be identically equal to zero.

The example cited shows that the statement of the problem on the origin of spontaneous rotation of axisymmetric flows in the presence of an electromagnetic field requires further study and more precise definition.

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